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About a fast isogeometric boundary element method

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(joint work with Jürgen Dölz, Michael Multerer, Stefan Kurz, Sebastian Schöps, and Felix Wolf)

1. Introduction

In the search of a method incorporating simulation techniques into the design workflow of industrial development, [8] proposed the concept of *Isogeometric Analysis (IGA)* to unite *Computer Aided Design (CAD)* and *Finite Element Analysis (FEA)*. It enables to perform simulations directly on geometries described by volumetric NURBS parametrizations. Nonetheless, many CAD systems use boundary representations only. Thus, volumetric parametrizations often have to be constructed solely for the purpose of simulation. The boundary parametrization, however, can be easily exported from CAD. Therefore, an approach via isogeometric boundary element methods seems to be natural.

2. Isogeometric boundary element methods

The utilization of parametric mappings in numerical implementations of the boundary element method is not new, going back further than the introduction of the isogeometric concept, see [6] for example. Parametric mappings avoid the problem of a slow convergence of the geometry due to the limited polynomial approximation of the geometry. Thus, they encourage the application of higher order Galerkin schemes. Through the parametric mappings, a tensor product structure on the geometries is induced, making it possible to define patchwise tensor product B-spline bases of high order and regularity.

One of the major downsides of the application of boundary element methods is that the integral operators involved yield dense discrete systems. To counteract the dense matrices, so-called *fast methods* must be employed for compression and efficiency. As shown in [3, 7], the tensor product structure induced by the mappings can be exploited to achieve an efficient implementation of compression techniques such as $\mathcal{H}$-matrices or the fast multipole method [5]. An isogeometric boundary element method promises hence runtimes which can compete with classical discretization methods.

Therefore, we developed the software library Bembel, **B**oundary **E**lement **Me**thod **B**ased **E**ngineering **L**ibrary, which is written in C and C++ [1]. It solves boundary value problems governed by the Laplace, Helmholtz or electric wave equation within the isogeometric framework. The development of the software started in the context of *wavelet Galerkin methods* on parametric surfaces, see [6], where the integration routines for the Green's function of the Laplacian have been developed and implemented. It was then extended to *hierarchical matrices* ($\mathcal{H}$-matrices) in [7] and to $\mathcal{H}^2$*-matrices* and higher order B-splines in [3]. With support of B-splines and NURBS for the geometry mappings, the Laplace and Helmholtz code became isogeometric in [2]. Finally, in [4], it has been extended to the electric field integral equation.

3. Numerical example

We shall present numerical results for the Laplace equation $\Delta U = 0$ inside the gear worm geometry Ω found in the left plot of Figure 1, whose surface $\Gamma = \partial\Omega$ is represented by 290 patches. The harmonic polynomial $U(\mathbf{x}) = 4x_1^2 - 3x_2^2 - x_3^2$ is used to prescribe either Dirichlet boundary conditions $f = U|_\Gamma$ or Neumann boundary conditions $g = \langle \nabla U, \mathbf{n}\rangle$ on Γ.

Making the single layer potential ansatz

$$U(\mathbf{x}) = \int_\Gamma \frac{u(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|_2} \mathrm{d}\sigma_{\mathbf{y}}, \quad \mathbf{x} \in \Omega, \tag{1}$$

leads to a Fredholm integral equation of the first kind

$$\mathcal{S}u(\mathbf{x}) = \int_\Gamma \frac{u(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|_2} \mathrm{d}\sigma_{\mathbf{y}} = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{2}$$

for the unknown density u in case of the Dirichlet problem. Whereas, making a double layer potential ansatz

$$U(\mathbf{x}) = \int_\Gamma \frac{\langle \mathbf{x}-\mathbf{y}, \mathbf{n}_{\mathbf{y}}\rangle u(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|_2^3} \mathrm{d}\sigma_{\mathbf{y}}, \quad \mathbf{x} \in \Omega, \tag{3}$$

amounts to a Fredholm integral equation of the first kind

$$\mathcal{W}u(\mathbf{x}) = \frac{\partial}{\partial \mathbf{n}_{\mathbf{x}}} \int_\Gamma \frac{\langle \mathbf{x}-\mathbf{y}, \mathbf{n}_{\mathbf{y}}\rangle u(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|_2^3} \mathrm{d}\sigma_{\mathbf{y}} = g(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{4}$$

for the unknown density u in case of the Neumann problem.

Since the density u is unknown, the error of the potential U is measured on the 115'241 vertices of a grid of 83'437 cubes fitted inside the domain. A visualization of these cubes together with the computed potential for the single layer ansatz can be found in the right plot of Figure 1. In view of having only a Lipschitz continuous boundary, the theoretical convergence rates are limited to at most h^3 for the single layer ansatz and to h^1 for the hypersingular ansatz. Figure 2 illustrates that these convergence rates are achieved for all polynomial degrees under consideration. In fact, the higher order ansatz functions even seem to produce a convergence rate of up to h^5 for both, the single layer ansatz and the hypersingular ansatz. Note that the dashed lines correspond to the convergence rates h^3 and h^5 while the

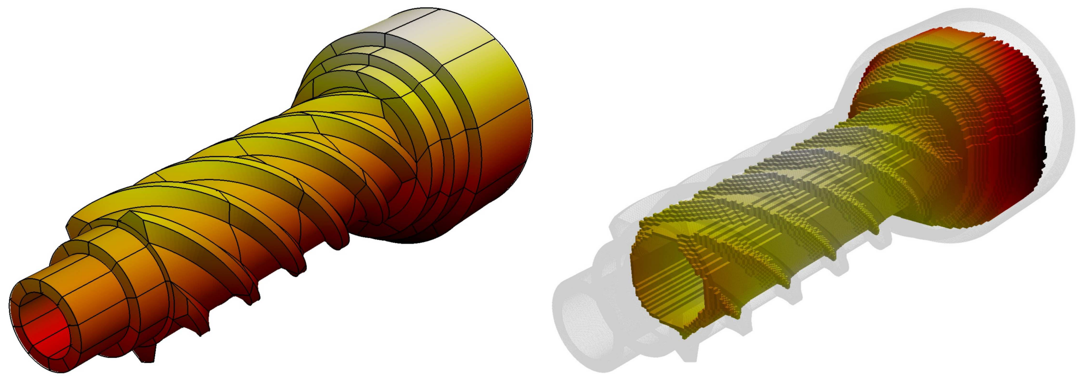

FIGURE 1. The gear worm geometry and the approximate potential in case of the Dirichlet problem for the Laplacian vs. level of uniform refinement.

accompanying numbers are the polynomial degrees of the interpolation in the fast multipole method.

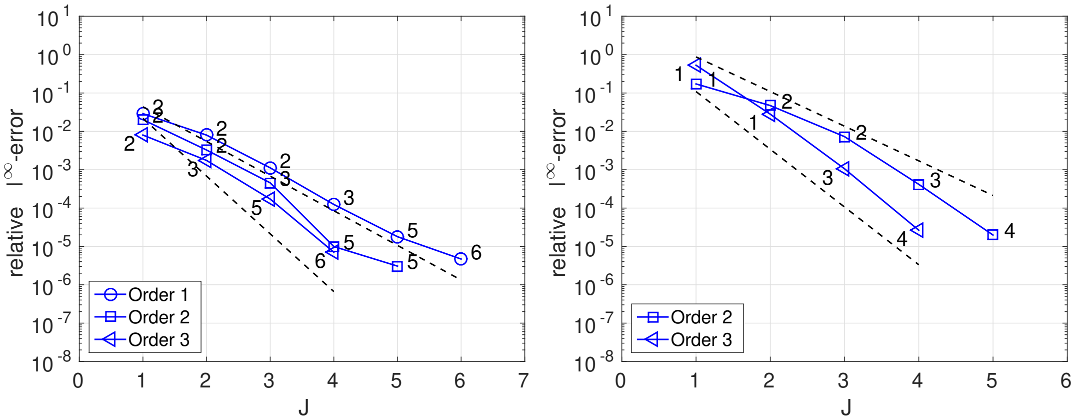

FIGURE 2. Relative errors of the potentials in case of the single layer ansatz (*left*) and the hypersingular ansatz (*right*).

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Mathematics of isogeometric analysis and applications: a status report

Thomas J.R. Hughes

I presented a sampling of the state-of-the-art in Isogeometric Analysis (IGA) with emphasis on mathematical developments. The field has become so enormous and broad based that it is impossible to even briefly mention all areas of activity. This is our second Oberwolfach workshop on IGA. The first was held in February of 2016 and was smaller than the present one. In my talk I tried to identify progress that has been made in the almost 3 ½ years since the first workshop. During that time, according to Web of Science, there have been approximately 1300 papers published on IGA in archival research journals. I began my talk with a comparison of the publication history of the first 30 years of the Finite Element Method (FEM) with that of IGA, which began in 2005. It is striking how quickly publications and citations in IGA have grown. In the first 10 years of IGA the numbers are much larger than in the first 30 years of FEM.

I presented a few applications with the FEM, specifically, automobile crash dynamics, full-body patient-specific, fluid-structure analysis of the cardiovascular system, and a HeartFlow, Inc., analsysi of blood flow in human coronary arteries. These illustrate the breadth and success of the FEM. I know a lot about all the applications because I developed many of the technologies employed. I observed that in each case the lowest order finite elements were utilized. Why not higher-order finite elements? From the academic research literature one would think the higher-order elements exhibit superior accuracy and efficiency. An academic answer might be that complex practical problems do not enjoy the solution regularity necessary to obtain higher-order convergence rates, but that is only a small part of the reason. The sad truth is that higher-order C^0-continuous finite elements are not robust and fail in many practical applications. Later in my talk, I used spectral analysis to reveal why this is the case and at the same time why spline-based approaches, such as IGA, do not suffer the same deficiencies. Indeed, one can show that maximally smooth C^{p-1}-continuous smooth splines exhibit a unique combination of accuracy and robustness. In fact, the higher the p, the more robust, the exact opposite of C^0-continuous finite elements.

Here are the main topics I covered in the rest of my talk: Basics technologies, such as B-splines and NURBS; approximation estimates in Sobolev norms; error analysis by spectral analysis techniques; Kolmogorov n-widths; efficient formation